

Precise Formulation of Neutrino Oscillation in the Earth

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Abstract

We give a perturbation theory of neutrino oscillation in the Earth. The perturbation theory is valid for neutrinos with energy $E \gtrsim 0.5$ GeV. It is formulated using trajectory dependent average potential. Non-adiabatic contributions are included as the first order effects in the perturbation theory. We analyze neutrino oscillation with standard matter effect and with non-standard matter effect. In a three flavor analysis we show that the perturbation theory gives a precise description of neutrino conversion in the Earth. Effect of the Earth matter is substantially simplified in this formulation.

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1 Introduction

One of the major concerns in neutrino oscillation experiments is the effect of Earth matter in neutrino flavor conversion [1, 2]. Because of the complicated matter density profile in the Earth it is a challenge to find a simple formula which can describe precisely neutrino conversion in the Earth. We rely a lot on numerical computation which does not offer us enough insights. Few years ago, an analytic and precise formulation was obtained for solar neutrinos, i.e. for $E \lesssim 30$ MeV [3, 4]. It is the purpose of the present paper to find a good formulation for higher energy neutrinos, i.e. for $E \gtrsim 0.5$ GeV. Previous works on neutrinos of this energy range include [5, 6, 7, 8, 9, 10, 11, 12, 13].

It will be shown that the approach using trajectory dependent average potential is very attractive. This approach was used in [14] in discussing solar neutrinos. It was used in [15] in a 2ν analysis for higher energy neutrinos ($E \gtrsim 0.5$ GeV). In the present paper we present a perturbation theory for three flavors of neutrinos. The perturbation theory uses trajectory dependent average potential. We do perturbative expansion using the deviation of potential around the average. We analyze the perturbation theory using the density profile of the Preliminary Earth Model (PREM) [16]. We study the perturbation

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theory for the case of standard matter effect and the case of non-standard matter effect with non-standard Neutral Current (NC) interactions. We show that the perturbation theory works very well for both cases. The theory is valid for neutrinos with $E \gtrsim 0.5$ GeV.

The perturbation theory presented is very useful for long baseline neutrino experiments [17, 18, 19, 20, 21]. It says for a fixed baseline ($\lesssim 6000$ km) the standard matter effect is a one parameter fit. Non-standard matter effect is also greatly simplified. The perturbation theory is of general interests to other types of neutrino sources, e.g. for studying atmospheric neutrinos, cosmic neutrinos from the galactic or extra-galactic sources. The present paper is organized as follows. In section 2 we present the perturbation theory of three flavor of neutrinos. In section 3 we study neutrino oscillation in the case of standard matter effect. In section 4 we extend the discussion on neutrino oscillation to the case of non-standard matter effect with non-standard neutrino NC interaction. We discuss and summarize in section 5.

2 Formulation of ν oscillation in the Earth

We consider oscillation of three flavors of neutrinos: $\psi = (\nu_e, \nu_\mu, \nu_\tau)$. The evolution equation is

$$i \frac{d}{dx} \psi(x) = H(x) \psi(x), \quad H(x) = H_0 + V(x) \quad (1)$$

$$H_0 = \frac{1}{2E} U \text{diag}\{0, \Delta m_{21}^2, \Delta m_{31}^2\} U^\dagger, \quad (2)$$

where $V(x)$, a 3×3 matrix, is the potential term accounting for the matter effect. $V(x)$ takes different form if neutrino interaction is different. U is the 3×3 neutrino mixing matrix in vacuum. U is parameterized using standard parameters θ_{12} , θ_{13} , θ_{23} and δ_{13} , the CP violating phase.

In solving the problem of neutrino oscillation we introduce the average potential term \bar{V} :

$$\bar{V} = \frac{1}{L} \int_0^L dx V(x), \quad (3)$$

where L is the length of neutrino trajectory in the Earth. Note that \bar{V} depends on the trajectory of neutrino in the Earth. It is averaged over potentials along the trajectory and is not averaged over potentials of all points in the Earth. Using \bar{V} we introduce \bar{H} and the mixing matrix U_m in matter which diagonalizes \bar{H} :

$$\bar{H} = H_0 + \bar{V}, \quad (4)$$

$$\bar{H} U_m = U_m \frac{1}{2E} \Delta, \quad \Delta = \text{diag}\{\Delta^1, \Delta^2, \Delta^3\}. \quad (5)$$

$\frac{1}{2E}\Delta^i (i = 1, 2, 3)$ are three eigenvalues of \bar{H} . $\Delta^1 < \Delta^2 < \Delta^3$ is satisfied in our convention. U_m is parameterized using parameters $\theta_{12}^m, \theta_{13}^m, \theta_{23}^m$ and the CP violating phase δ_{13}^m . Note that we work in the convention of Hamiltonian introduced in (2) and (4). In typical cases where 3×3 evolution problem can be reduced to 2×2 evolution problem, e.g. when hierarchy in eigenvalues are present, eigenvalues obtained in our convention can be related to eigenvalues solved in standard 2×2 problem by shifting the phase of neutrinos and making the reduced hamiltonian traceless.

We are ready to solve the evolution problem in (1). Note that $H(x)$ can be re-written as

$$H(x) = \bar{H} + \delta V(x), \quad \delta V(x) = V(x) - \bar{V}. \quad (6)$$

We first solve the evolution by \bar{H} and obtain the contribution of δV using perturbation in δV . Keeping result of first order in δV we obtain

$$\psi(L) = M(L)\psi(0), \quad (7)$$

$$M(L) = U_m e^{-i\frac{\Delta}{2E}L} (1 - iC) U_m^\dagger, \quad (8)$$

where C is a 3×3 matrix accounting for the non-adiabatic transition:

$$C = \int_0^L dx e^{i\frac{\Delta}{2E}x} U_m^\dagger \delta V(x) U_m e^{-i\frac{\Delta(x)}{2E}x}. \quad (9)$$

It is clear that $C^\dagger = C$ holds. One can see that

$$C_{jj} = \int_0^L dx (U_m^\dagger \delta V(x) U_m)_{jj} = 0, \quad (10)$$

$$C_{jk} = \int_0^L dx e^{i\frac{\Delta^j - \Delta^k}{2E}x} (U_m^\dagger \delta V(x) U_m)_{jk}, \quad j \neq k. \quad (11)$$

Eq. (10) is guaranteed by Eq. (3). $|C_{jk}| \ll 1 (j \neq k)$ should be satisfied if this is a good perturbation theory. One of the virtues of this perturbation theory is that Eq. (10) guarantees that the oscillation phase is correctly reproduced.

In the next two sections we will discuss in detail that we indeed do expansion in small quantities when computing $C_{jk} (j \neq k)$ and correction of the second order $\mathcal{O}(C^2)$ is further suppressed by these small quantities. Hence we can convince ourselves that we are dealing with a perturbation theory. Before making detailed discussions on it we note that it is a perturbation partly because density changes mildly in the mantle or in the core of the Earth. In the mantle or in the core $\|\delta V\|/\|V\| \lesssim 0.3$. This helps in improving the perturbation theory.

When $L > 10690$ km neutrinos cross the core of the Earth. Large density jump between the core and the mantle can cause problem to the simplest version of the perturbation theory. To improve the approximation we use average potentials for parts of trajectory in the core and in the mantle separately. So the evolution matrix can be written as

$$M = M_3 M_2 M_1, \quad (12)$$

where M_2 is the evolution matrix in the core and $M_{1,3}$ are evolution matrices in the mantle. They are

$$M_i = U_{mi} e^{-i \frac{\Delta_i}{2E} (L_i - L_{i-1})} (1 - i C_i) U_{mi}^\dagger, \quad i = 1, 2, 3 \quad (13)$$

where $L_0 = 0$ and $L_3 = L$. $x = L_1$ is the point where neutrinos cross from the mantle to the core. $x = L_2$ is the point where neutrinos come out of the core to the mantle. Δ_i and U_{mi} are the eigenvalues and mixing matrix in the region $x \in [L_{i-1}, L_i]$. They are computed using average potential

$$\bar{V}_i = \frac{1}{L_i - L_{i-1}} \int_{L_{i-1}}^{L_i} dx V(x). \quad (14)$$

C_i is computed using Eq. (9) in the region $x \in [L_{i-1}, L_i]$. Because of the approximate symmetric density profile of the Earth we have

$$\bar{V}_3 \approx \bar{V}_1.$$

This strategy to treat evolution of core crossing neutrinos was also noticed in Ref. [15]. In the present paper when we do computation using the perturbation theory we always use Eq. (12) for core crossing trajectories which have $L > 10690$ km and use Eq. (8) for trajectories crossing only mantle which have $L < 10690$ km.

For anti-neutrinos the hamiltonian is

$$H'(x) = H_0^T - V^T(x), \quad (15)$$

where $V^T(x)$ is the transpose of the potential term $V(x)$ introduced in Eq. (1). H_0^T is the transpose of H_0 . The problem of neutrino evolution can be solved using average potential, similar to the program used for neutrinos.

The probability of neutrino oscillation is introduced using the evolution matrix:

$$P(\nu_k \rightarrow \nu_l) = |M_{lk}(x = L)|^2, \quad P(\bar{\nu}_k \rightarrow \bar{\nu}_l) = |\bar{M}_{lk}(x = L)|^2, \quad (16)$$

where \bar{M}_{kl} is the evolution matrix of anti-neutrinos $\bar{\psi} = \text{diag}\{\bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau\}$:

$$\bar{\psi}(L) = \bar{M}(L) \bar{\psi}(0).$$

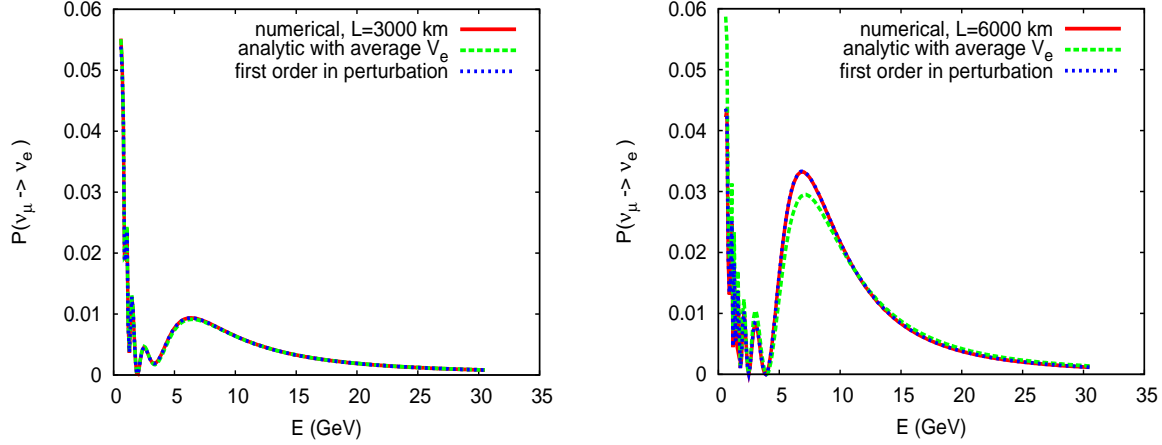


Figure 1: $P(\nu_\mu \rightarrow \nu_e)$ versus energy for the case with standard matter effect. Left panel is for $L = 3000$ km and right panel is for $L = 6000$ km. $\Delta m_{21}^2 = 8. \times 10^{-5}$ eV², $\Delta m_{32}^2 = 3. \times 10^{-3}$ eV². $\sin^2 2\theta_{23} = 1$, $\tan^2 \theta_{12} = 0.41$, $\sin^2 2\theta_{13} = 0.01$, $\delta_{13} = \pi/6$. PREM density profile is used for computation in this figure and all remaining figures in this paper.

Using the oscillation probability we can define observables of CP violation and of time reversal asymmetry. For example we define

$$A_{CP} = \frac{P(\nu_\mu \rightarrow \nu_\tau) - P(\bar{\nu}_\mu \rightarrow \bar{\nu}_\tau)}{P(\nu_\mu \rightarrow \nu_\tau) + P(\bar{\nu}_\mu \rightarrow \bar{\nu}_\tau)}, \quad (17)$$

$$A_T = \frac{P(\nu_e \rightarrow \nu_\mu) - P(\nu_\mu \rightarrow \nu_e)}{P(\nu_e \rightarrow \nu_\mu) + P(\nu_\mu \rightarrow \nu_e)}, \quad (18)$$

One can see in Eqs. (1) and (15) that A_{CP} does not contain pure information of fundamental CP violating parameter. Because the medium contains only matter but not anti-matter A_{CP} is not zero even if CP violating phase δ_{13} in U is zero. It is clear that powerful electron neutrino beam is needed to observe the time reversal asymmetry. Intense electron neutrino beam is probably available using muon storage technology in future experiments.

3 Oscillation with standard matter effect

In this section we consider oscillation of neutrinos with standard matter effect. In this case we express the potential, mixing matrix, etc as

$$V(x) = V_s(x), \quad \bar{V} = \bar{V}_s, \quad H = H_s, \quad \bar{H} = \bar{H}_s, \quad U_m = U_s, \quad \Delta = \Delta_s. \quad (19)$$

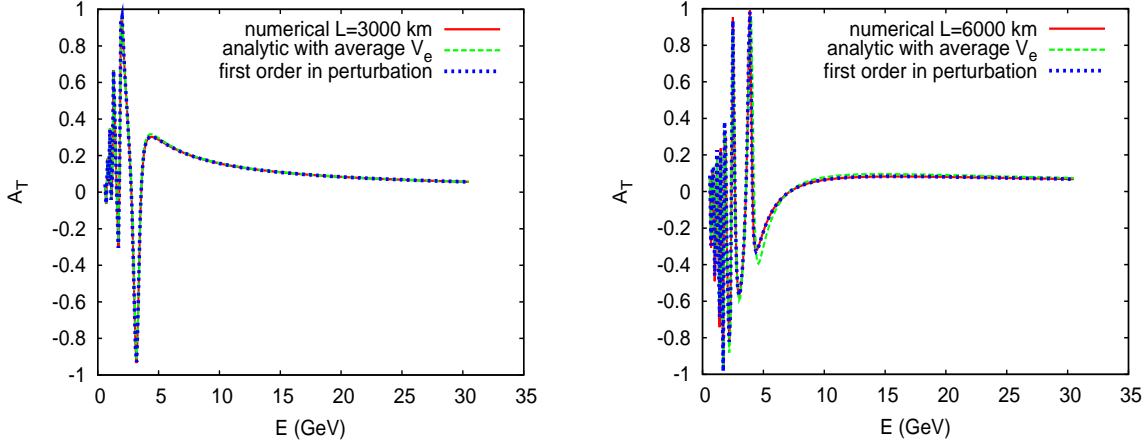


Figure 2: A_T versus energy for the case with standard matter effect. Left panel is for $L = 3000$ km and right panel is for $L = 6000$ km. Parameters are the same as in Fig. 1.

The potential term of standard matter effect is

$$V_s(x) = \text{diag}\{V_e(x), 0, 0\}. \quad (20)$$

$V_e = \sqrt{2}G_F N_e$ is the potential in matter. N_e is the electron number density in matter. G_F is the Fermi constant. Effect of standard NC interaction is universal for three types of neutrinos and is neglected for neutrino oscillation in Earth matter. \bar{V}_s , H_s , \bar{H}_s , U_s , Δ_s are defined using Eqs. (3), (1), (2), (4) and (5). C_{jk} is obtained as follows:

$$C_{jk} = \int_0^L dx e^{i\frac{\Delta_s^j - \Delta_s^k}{2E}x} (U_s)_{ej}^* (U_s)_{ek} \delta V_e(x), \quad j \neq k, \quad (21)$$

where

$$\delta V_e(x) = V_e(x) - \bar{V}_e, \quad \bar{V}_e = \frac{1}{L} \int_0^L dx V_e(x). \quad (22)$$

For trajectories crossing mantle only the evolution matrix is

$$M_s(L) = U_s e^{-i\frac{\Delta_s}{2E}L} (1 - iC) U_s^\dagger. \quad (23)$$

For core crossing trajectories ($L > 10690$ km) the evolution matrix is generalized using Eq. (12).

$$M_s(L) = M_{s3} M_{s2} M_{s1}. \quad (24)$$

M_{s2} and $M_{s1,s2}$ are evolution matrices in the core and mantle separately. They are computed to the first order in δV and using average potentials in the core and the mantle separately, as explained in section 2.

For comparison we also show the results computed using analytic formula:

$$M_{As}(L) = U_s e^{-i\frac{\Delta s}{2E}L} U_s^\dagger, \quad (25)$$

where non-adiabatic contribution is not included. This formula is not generalized for core crossing trajectories as done in Eq. (24).

In Fig. 1 we plot $P(\nu_\mu \rightarrow \nu_e)$ versus energy. We show results of numerical computation, results computed using Eq. (23) and the result computed using analytic formula Eq. (25). The lines labeled with "analytic with average V_e " are computed using Eq. (25). The lines labeled with "first order in perturbation" are computed using Eq. (23). They are plots for $L = 3000$ km and 6000 km. One can see that results computed using the perturbation theory reproduce precisely the phase and the magnitude of the oscillation pattern. One can see that for these two baselines the analytic results computed using Eq. (25) give a quite good approximation to the oscillation pattern. Actually one can not see difference in the left panel ($L = 3000$ km). In the right panel ($L = 6000$ km) one can see that oscillation phase is correctly reproduced by the analytic result but the magnitude around peaks is not precisely reproduced. There are some small differences.

In the right panel height of the first peak of the right side is given by $\sin^2 2\theta_{13}^m$. One can see that it is much larger than the magnitude given by the vacuum value $\sin^2 2\theta_{13}$. This is because the position of the first peak is close to the region of $1 - 3$ MSW resonance which has energy range $E \sim 7 - 10$ GeV. Magnitudes of the second and third peaks are much smaller. This is because the second and third peaks are away from the resonance region and the vacuum value is dominant. In Fig. 2 we show plots for the time reversal asymmetry A_T versus energy. Again in the left panel the analytic result using Eq. (25) is precise. In the right panel there are some small differences between numerical and analytic results. Results computed using the perturbation theory, i.e. using Eq. (23), are in remarkable agreement with the numerical results.

The analytic formula is a better approximation for shorter baseline. In Figs. (3) and (4) we plot $P(\nu_\mu \rightarrow \nu_e)$ versus L , the length of baseline. For $L > 10690$ km the lines labeled with "first order in perturbation" are computed using (24). Left panels in Figs. (3) and (4) are for $E = 5$ GeV and right panels are for $E = 15$ GeV. One can see clearly that the analytic result is a very good approximation for $L \lesssim 6000$ km. It's no longer precise for $L \gtrsim 6000$ km. In all these plots the results computed using the perturbation theory are always in remarkable agreement with that of numerical computation.

In Fig. 5 and 6 we also compare computations on $P(\nu_\mu \rightarrow \nu_\tau)$ and $P(\nu_e \rightarrow \nu_\tau)$. We see that $P(\nu_\mu \rightarrow \nu_\tau)$ is very well approximated by the analytic result computed using Eq. (25) except for $L \gtrsim 11500$ km. For $P(\nu_e \rightarrow \nu_\tau)$ there are some differences for $L \gtrsim 6000$

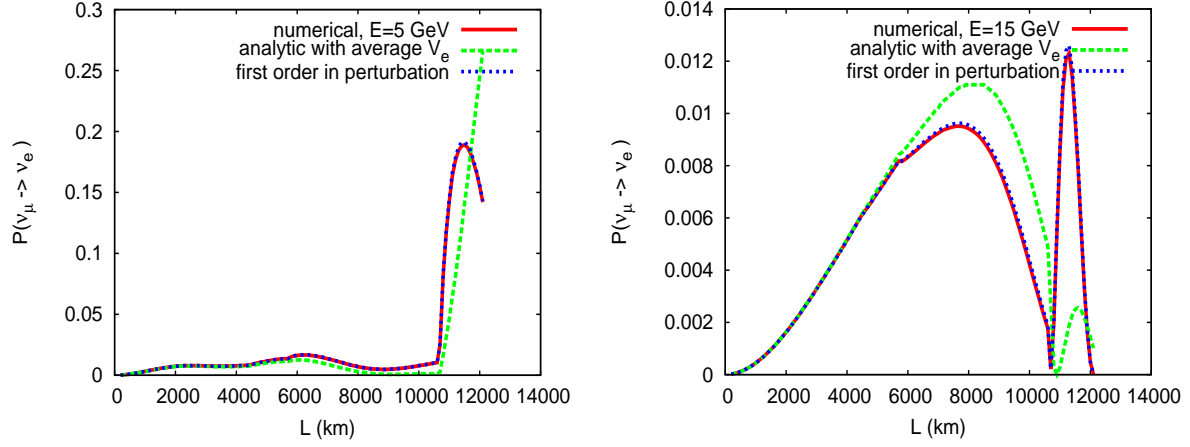


Figure 3: $P(\nu_\mu \rightarrow \nu_e)$ versus L the length of baseline for the case with standard matter effect. Left panel is for $E = 5$ GeV and right panel is for $E = 15$ GeV. Neutrino parameters are the same as in Fig. 1.

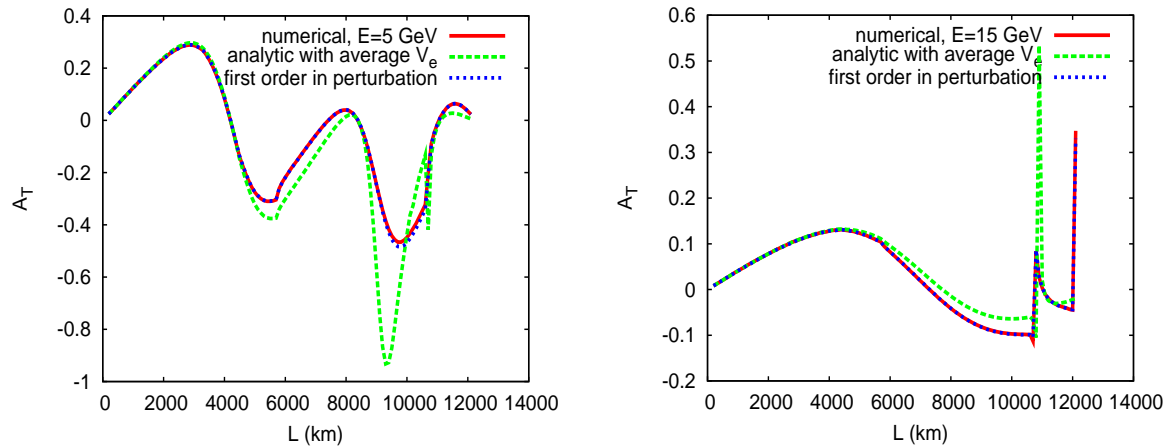


Figure 4: A_T versus L the length of baseline for the case with standard matter effect. Left panel is for $E = 5$ GeV and right panel is for $E = 15$ GeV. Neutrino parameters are the same as in Fig. 1.

km. Differences of these magnitude are negligible in $P(\nu_\mu \rightarrow \nu_\tau)$. Again we see that the results of first order in the perturbation theory are in remarkable agreement with the results of numerical computation.

One can understand the success of this formulation of neutrino oscillation by noting that indeed we are expanding in small quantities and we are dealing with a perturbation theory. In the following we show that $C_{jk}(j \neq k)$ computed using Eq. (21) is suppressed by some small quantities. Hence in $(1 - iC)$ condition

$$|C_{jk}| \ll 1 \quad (j \neq k) \quad (26)$$

can be satisfied in the first order. The second order result of order $\mathcal{O}(C^2)$ is further suppressed. Thus we can be confident on the perturbation theory. Note that another condition is that density changes mildly in the mantle or in the core, as already emphasized in section (2). (26) is explained as follows. We first consider the case with $\Delta m_{31}^2 > 0$.

i) For $0.5 \text{ GeV} \ll E < 7 - 10 \text{ GeV}$, $\frac{\Delta m_{21}^2}{2E} \ll \bar{V}_e < \frac{\Delta m_{31}^2}{2E}$ holds and this is the region below 1–3 MSW resonance. In this range of energy the eigenvalues of \bar{H} are approximately $(\frac{\Delta m_{21}^2}{2E} \cos^2 \theta_{12}, \bar{V}_e, \frac{\Delta m_{31}^2}{2E})$ in the limit that correction of $\mathcal{O}(\sin \theta_{13})$ and $\mathcal{O}(\Delta m_{21}^2/(2E\bar{V}_e))$ are neglected. Small correction of $\mathcal{O}(\frac{\Delta m_{21}^2}{2E})$ has been neglected in larger eigenvalues. Hence the first entries of \bar{H} are changed to the second entries. We have

$$\sin 2\theta_{13}^m \approx \sin 2\theta_{13}, \quad \cos \theta_{12}^m \approx \frac{\Delta m_{21}^2}{4E} \frac{1}{\bar{V}_e} \sin 2\theta_{12}. \quad (27)$$

So we get $|(U_s)_{e3}| = \sin \theta_{13}^m \ll 1$ and $(U_s)_{e1} = \cos \theta_{12}^m \cos \theta_{13}^m \ll 1$. $C_{jk} \propto (U_s)_{ej}^* (U_s)_{ek} (j \neq k)$ is suppressed either by $(U_s)_{e3}$ or by $(U_s)_{e1}$.

In this range of energy C_{jk} is suppressed by small quantities $\sin \theta_{13}$ and $\Delta m_{21}^2/(2E\bar{V}_e)$.

ii) For $E > 7 - 10 \text{ GeV}$, $\bar{V}_e > \frac{\Delta m_{31}^2}{2E}$ holds and this is the region above 1–3 MSW resonance. In this range of energy the eigenvalues of \bar{H} are approximately $(\frac{\Delta m_{21}^2}{2E} \cos^2 \theta_{12}, \frac{\Delta m_{31}^2}{2E}, \bar{V}_e)$ in the limit that corrections of $\mathcal{O}(\sin \theta_{13})$ and $\mathcal{O}(\Delta m_{21}^2/(2E\bar{V}_e))$ are neglected. In this case the first entries of \bar{H} are changed to the third entries. We can get (see appendix)

$$\cos \theta_{13}^m \approx \frac{\Delta m_{31}^2}{4E} \frac{1}{\bar{V}_e} \sin 2\theta_{13} \ll 1. \quad (28)$$

So we have $(U_s)_{e1} = \cos \theta_{13}^m \cos \theta_{12}^m \ll 1$ and $(U_s)_{e2} = \cos \theta_{13}^m \sin \theta_{12}^m \ll 1$. C_{jk} is suppressed either by $(U_s)_{e1}$ or by $(U_s)_{e2}$.

In this range of energy C_{jk} is suppressed by small quantity $\Delta m_{31}^2/(2E\bar{V}_e) \sin 2\theta_{13}$.

iii) For $E \sim 7 - 10 \text{ GeV}$, condition $\bar{V}_e = \frac{\Delta m_{31}^2}{2E} \cos 2\theta_{13}$ can be satisfied and the MSW resonance can happen. In this case $\cos \theta_{12}^m$ is expressed using Eq. (27). C_{12} and C_{13} are

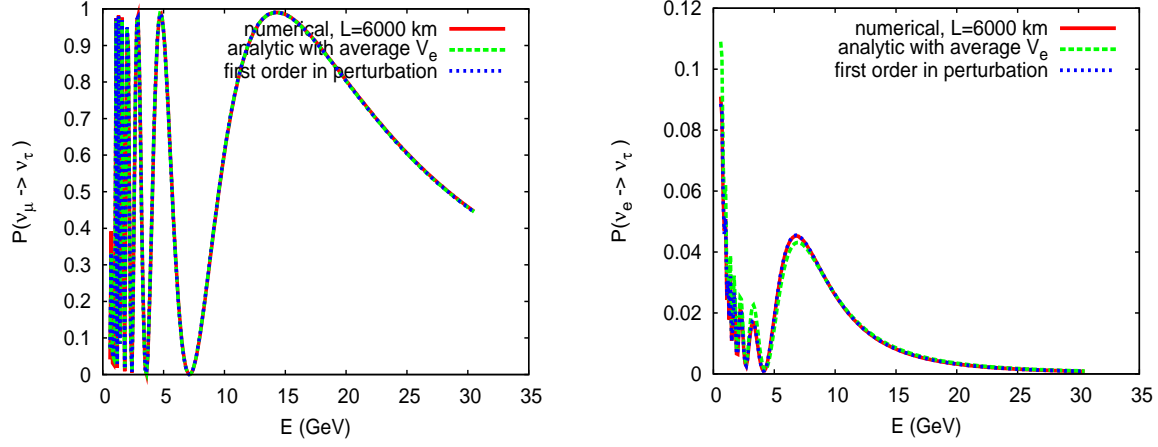


Figure 5: Plots for the case with standard matter effect. Left panel, $P(\nu_\mu \rightarrow \nu_\tau)$ versus energy, $L = 6000$ km; right panel, $P(\nu_e \rightarrow \nu_\tau)$ versus energy, $L = 6000$ km. Other parameters are given in Fig. 1.

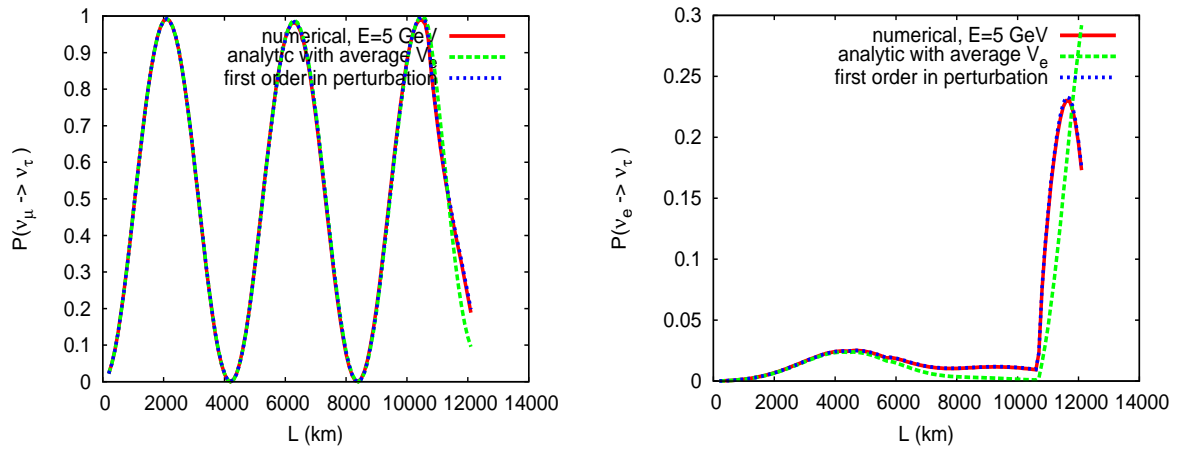


Figure 6: Plots for the case with standard matter effect. Left panel, $P(\nu_\mu \rightarrow \nu_\tau)$ versus L , $E = 5$ GeV; right panel, $P(\nu_e \rightarrow \nu_\tau)$ versus L , $E = 5$ GeV. Other parameters are given in Fig. 1.

suppressed by it. But C_{23} is no longer suppressed by $\sin \theta_{13}^m$ or $\cos \theta_{13}^m$ since $\theta_{13}^m \approx \pi/4$ if resonance happens.

One can see that C_{23} is suppressed by the resonance condition itself. In the resonance region the second and third mass eigenstates get almost degenerate:

$$\frac{1}{2E}(\Delta_s^3 - \Delta_s^2) \approx \frac{\Delta m_{31}^2}{2E} \sin 2\theta_{13} + \mathcal{O}\left(\frac{\Delta m_{21}^2}{2E}\right). \quad (29)$$

It is clear that in the resonance region

$$\phi(x) = \frac{\Delta_s^3 - \Delta_s^2}{2E}(x - L/2)$$

is a small number. So

$$\begin{aligned} C_{23} &= e^{-i\frac{\Delta_s^3 - \Delta_s^2}{4E}L} \int_0^L dx e^{-i\phi(x)} (U_s)_{e2}^* (U_s)_{e3} \delta V_e(x) \\ &= e^{-i\frac{\Delta_s^3 - \Delta_s^2}{4E}L} \int_0^L dx (U_s)_{e2}^* (U_s)_{e3} \delta V_e(x) \left[1 - i\phi(x) - \frac{1}{2}\phi^2(x) + \dots\right]. \end{aligned} \quad (30)$$

The first term in the bracket gives zero after integration. The second term gives zero after taking into account the fact that δV is approximately symmetric in the Earth. Since major contribution is from ϕ^2 term it is not hard to see that $|C_{23}|$ is indeed suppressed by small numbers.

In this range of energy C_{jk} is suppressed by $\Delta m_{21}^2/(2E\bar{V}_e)$ and $\Delta m_{31}^2 L/(2E) \sin 2\theta_{13}$.

If $\Delta m_{31}^2 < 0$, eigenvalues of \bar{H} are roughly $(\frac{\Delta m_{31}^2}{2E}, \frac{\Delta m_{21}^2}{2E} \cos^2 \theta_{12}, \bar{V}_e)$. It is obtained by interchanging the first and third entries of \bar{H} . Similar to the discussion on point ii) we can get $\cos \theta_{13}^m \approx \sin 2\theta_{13} |\Delta m_{31}^2|/(2|\Delta m_{31}^2| + 4E\bar{V}_e)$. It is easy to see that we are expanding C_{jk} using small quantities $(U_s)_{ej}^* (U_s)_{ek} (j \neq k)$ which is suppressed by $\cos \theta_{13}^m$.

For anti-neutrino the MSW resonance happens for $\Delta m_{31}^2 < 0$. Discussions on the perturbation theory closely follow the discussions for neutrinos above. Similarly, one can show that $C_{jk} (j \neq k)$ is expanded using small quantities. The second order of perturbation theory is of order $\mathcal{O}(C^2)$. Hence it is further suppressed.

We summarize that we do expansion using quantities $|(U_s)_{ej}^* (U_s)_{ek}| (j \neq k)$ in the perturbation theory. Except in the MSW resonance region these quantities are small in energy range $E \gtrsim 0.5$ GeV. In the resonance region the resonance condition itself guarantees the validness of the perturbative expansion.

4 Oscillation with non-standard matter effect

In this section we extend the discussion to case with non-standard matter effect. Previous works on non-standard matter effect include Refs. [24, 25]. In this case we express the

potential, mixing matrix, etc as

$$V(x) = V_n(x), \quad \bar{V} = \bar{V}_n, \quad H = H_n, \quad \bar{H} = \bar{H}_n, \quad U_m = U_n, \quad \Delta = \Delta_n. \quad (31)$$

Physics beyond the Standard Model can give non-standard four fermion interactions such as $\bar{q}\gamma_\mu q \bar{\nu}_k\gamma^\mu\nu_l$ and $\bar{e}\gamma_\mu e \bar{\nu}_k\gamma^\mu\nu_l$. Neutrino evolution in matter is modified by these terms. In general the potential can be written as follows

$$V_n(x) = \text{diag}\{V_e, 0, 0\} + \begin{pmatrix} 0 & V_{e\mu} & V_{e\tau} \\ V_{\mu e} & V_{\mu\mu} & V_{\mu\tau} \\ V_{\tau e} & V_{\tau\mu} & V_{\tau\tau} \end{pmatrix}, \quad (32)$$

where $V_e = \sqrt{2}G_F N_e$ is the potential with standard charged current interaction, V_{kl} is from non-standard NC interaction. $V_{lk}^* = V_{kl}$ because the Hamiltonian is hermitian. x dependence in V_{kl} has been suppressed in Eq. (32). V_{ee} has been made zero in our convention. This is achieved by shifting the phases of neutrinos: $\nu_l \rightarrow e^{-i \int dx V_{ee}} \nu_l$, $V_{kl} \rightarrow V_{kl} + V_{ee}$. In this convention V_{kl} is

$$\begin{aligned} V_{kl} &= \sqrt{2}G_F [(f_{kl} - f_{ee})N_e + (g_{kl} - g_{ee})N_p + (h_{kl} - h_{ee})N_n], \\ &= V_e [(f_{kl} - f_{ee} + g_{kl} - g_{ee}) + (h_{kl} - h_{ee})N_n/N_e], \end{aligned} \quad (33)$$

where f_{kl} , g_{kl} and h_{kl} are the dimensionless strengths of non-standard four Fermion interactions $\sqrt{2} f_{kl} G_F \bar{e}\gamma_\mu e \bar{\nu}_k\gamma^\mu\nu_l$, $\sqrt{2} g_{kl} G_F \bar{p}\gamma_\mu p \bar{\nu}_k\gamma^\mu\nu_l$ and $\sqrt{2} h_{kl} G_F \bar{n}\gamma_\mu n \bar{\nu}_k\gamma^\mu\nu_l$. N_p and N_n are number densities of proton and neutron in matter. The second line in Eq. (33) holds because of the equality $N_e = N_p$ in neutral matter.

We can re-write V_n as

$$V_n = V_e \begin{pmatrix} 1 & \epsilon_{e\mu} & \epsilon_{e\tau} \\ \epsilon_{\mu e} & \epsilon_{\mu\mu} & \epsilon_{\mu\tau} \\ \epsilon_{\tau e} & \epsilon_{\tau\mu} & \epsilon_{\tau\tau} \end{pmatrix}. \quad (34)$$

where $\epsilon_{kl} = V_{kl}/V_e$. As can be seen in Eq. (33), ϵ_{kl} is not a constant if N_n/N_e is not a constant in matter. This happens when chemical composition changes in matter. In the present paper we are not going to consider the possibility that ϵ_{kl} changes in the neutrino trajectory. We set ϵ_{kl} constant in the analysis. This possibility is achieved when non-standard interactions are from interactions of neutrino with proton or with electron.

Four fermion interactions of muon neutrinos are well constrained by direct test, e.g. by NuTeV experiment [22]. Since ϵ_{kl} is a coherent combination of strengths f_{kl} , g_{kl} and h_{kl} , constraints on the strengths of these couplings from collider or fixed target experiments can not be directly translated to constraints on ϵ_{kl} . However, one can have a rough

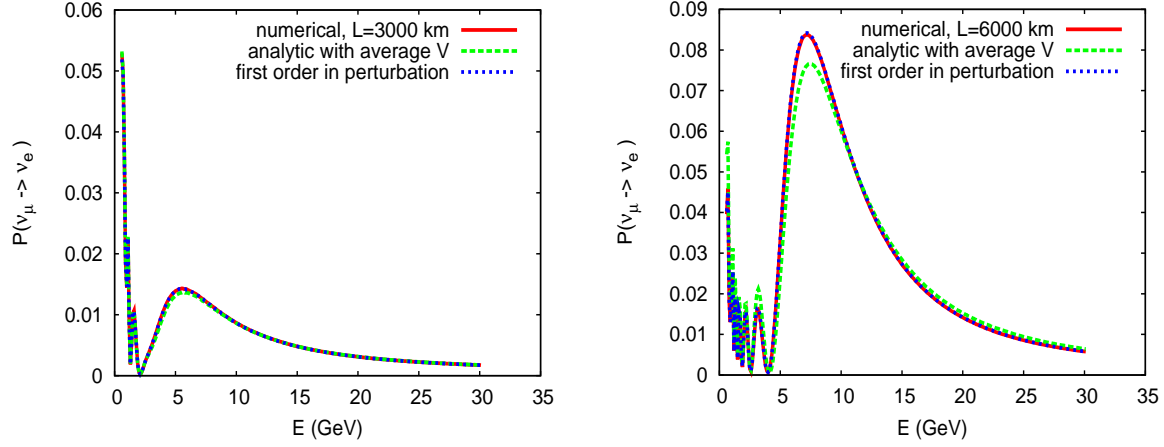


Figure 7: $P(\nu_\mu \rightarrow \nu_e)$ versus energy for the case with non-standard matter effect. Left panel is for $L = 3000$ km; right panel is for $L = 6000$ km. $\epsilon_{e\mu} = 0.01 e^{-i\pi/20}$, $\epsilon_{e\tau} = 0.04 e^{-i\pi/3}$, $\epsilon_{\mu\tau} = 0.01 e^{-i\pi/20}$. Other parameters are the same as in Fig. 1

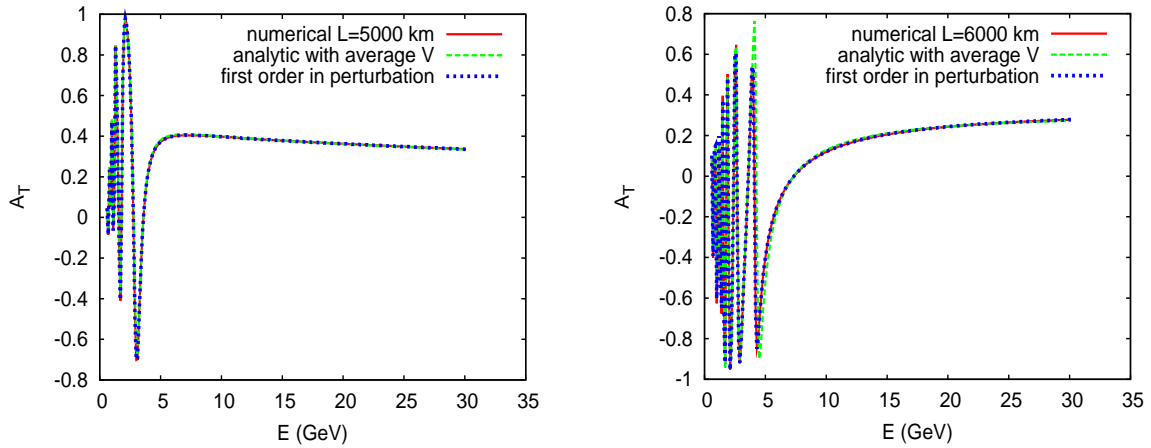


Figure 8: A_T versus energy for the case with non-standard matter effect. Left panel is for $L = 3000$ km and right panel is for $L = 6000$ km. $\epsilon_{e\tau} = 0.01 e^{-i\pi/20}$, $\epsilon_{e\mu} = 0.04 e^{-i\pi/3}$, $\epsilon_{\mu\tau} = 0.01 e^{-i\pi/20}$. Other parameters are the same as in Fig. 1.

constraint on the magnitude of ϵ_{kl} . The present experiments can reach precision of about one percent [23]. Hence, one can induce that $|\epsilon_{\mu e}|, |\epsilon_{\mu \tau}| \lesssim 10^{-2}$. $f_{ee, e\tau, \tau\tau}$, $g_{ee, e\tau, \tau\tau}$, $h_{ee, e\tau, \tau\tau}$ are not well constrained because there are no powerful electron neutrino and tau neutrino beams. [†] Hence in our convention $\epsilon_{e\tau}$, $\epsilon_{\mu\mu}$ and $\epsilon_{\tau\tau}$ are not well constrained by direct tests. Other constraints on ϵ_{kl} come from neutrino oscillation experiments. Previous studies[26, 27] show that $|\epsilon_{\mu\mu}|, |\epsilon_{\tau\tau}| \lesssim 10^{-2}$ and $|\epsilon_{e\tau}| \lesssim 10^{-1}$.

The trajectory dependent average potential and average Hamiltonian are

$$\bar{H}_n = H_0 + \bar{V}_n, \quad \bar{V}_n = \begin{pmatrix} \bar{V}_e & \bar{V}_{e\mu} & \bar{V}_{e\tau} \\ \bar{V}_{\mu e} & \bar{V}_{\mu\mu} & \bar{V}_{\mu\tau} \\ \bar{V}_{\tau e} & \bar{V}_{\tau\mu} & \bar{V}_{\tau\tau} \end{pmatrix}, \quad (35)$$

where \bar{V}_e is given in Eq. (22). \bar{V}_{kl} is similarly defined. Mixing matrix U_n diagonalizes \bar{H}_n , as done in Eq. (5). $\Delta = \Delta_n$ is obtained after diagonalization.

Keeping the first order result in δV the evolution matrix is solved as

$$M_n(x) = U_n e^{-i\frac{\Delta_n}{2E}x} (1 - iC) U_n^\dagger, \quad (36)$$

$$C_{jk} = \int_0^L dx e^{i\frac{\Delta_n^j - \Delta_n^k}{2E}x} [(U_n)_{ej}^* (U_n)_{ek} \delta V_e(x) + \sum_{s,t} (U_n)_{sj}^* (U_n)_{tk} \delta V_{st}(x)], \quad (37)$$

where $\delta V_{st} = V_{st} - \bar{V}_{st}$. $C_{jj} = 0$ holds as explained in section 2. For core crossing trajectories ($L > 10690$ km) the evolution matrix is generalized using Eq. (12) as follows

$$M_n(L) = M_{n3} M_{n2} M_{n1}. \quad (38)$$

where $M_{n1, n2}$ and M_{n3} are evolution matrices in the mantle and in the core. They are computed to the first order in δV and using average potentials in the core and in the mantle separately, as explained in section 2.

We also show result computed using analytic formula

$$M_{An}(L) = U_n e^{-i\frac{\Delta_n}{2E}L} U_n^\dagger. \quad (39)$$

This formula is not generalized for core crossing trajectories.

In Fig. 7 we plot $P(\nu_\mu \rightarrow \nu_e)$ versus energy for the case with non-standard matter effect. We show results of numerical computation, results computed using the perturbation theory and the results computed using the analytic formula Eq. (39). The left panel in the figure is for $L = 3000$ km and the right panel is for $L = 6000$ km. In the

[†]Intense electron neutrino or anti-neutrino source is available at low energy ($E \lesssim 30$ MeV) from stopped muon decay or from reactor. But they are not enough to make strong constraints.

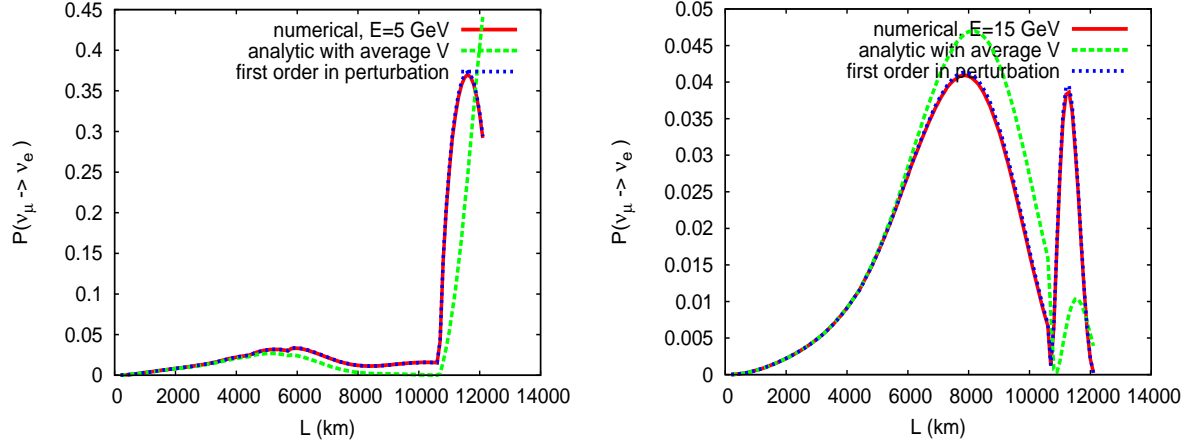


Figure 9: $P(\nu_\mu \rightarrow \nu_e)$ versus L the length of baseline for the case with non-standard matter effect. Left panel is for $E = 5$ GeV and right panel is for $E = 15$ GeV. Neutrino parameters are the same as in Fig. 1.

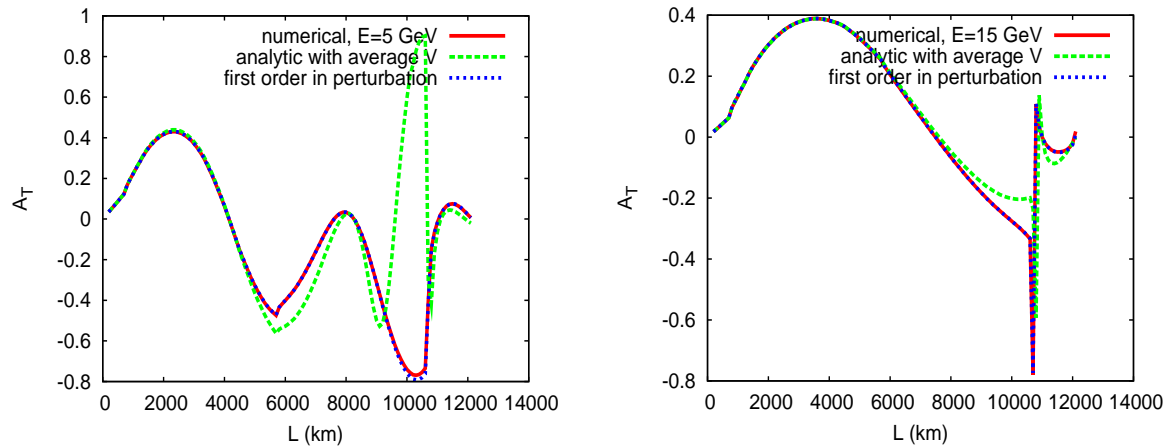


Figure 10: A_T versus L the length of baseline for the case with non-standard matter effect. Left panel is for $E = 5$ GeV and right panel is for $E = 15$ GeV. Neutrino parameters are the same as in Fig. 1.

right panel one can see clearly the effect of MSW resonance in the first peak from right hand side. $\sin^2 2\theta_{13}^m$ is enhanced to about 0.1, much larger than the vacuum value 0.01 chosen in the analysis. The results computed in the perturbation theory are in perfect agreement with that of the numerical computations. The analytic result is not always in perfect agreement with that of the numerical computation. In the right panel one can see some differences of the two computations. These differences can also be seen clearly in the right panel of Fig. 8 where the time reversal asymmetry is plotted.

In Fig. 9 we plot $P(\nu_\mu \rightarrow \nu_e)$ versus L the length of the baseline. In Fig. 10 the time reversal asymmetry is plotted. One can see clearly the remarkable agreement between the computation using the perturbation theory and the numerical computation. The analytic computation is a good approximation for $L \lesssim 6000$ km.

We can show that indeed we are expanding in small quantities and we are doing a perturbation theory. First, the second term in the bracket of (37) gives small contribution. This is because $|V_{kl}|/V_e \lesssim 0.1$. Hence $|\delta V_{kl}|L = (|\delta V_{kl}|/|V_{kl}|)|V_{kl}|L$ is a small number. Second, the discussion on the first term closely follows the discussions in the previous section. In lower energy region ($0.5 \text{ GeV} \lesssim E < 7 - 10 \text{ GeV}$) the discussion in the previous section is still valid since V_{kl} does not change the mixing matrix very much. For energy $E > 10 \text{ GeV}$, V_{kl} can be important. When V_{kl} is important enough to determine mixing matrix we have (see appendix)

$$\cos \theta_{13}^m \approx \sqrt{|V_{\tau e}|^2 + |V_{\mu e}|^2}/V_e \ll 1. \quad (40)$$

Contribution of the first term is suppressed by $(U_n)_{e1,e2}$ which are proportional to $\cos \theta_{13}^m$.

We summarize that in the presence of non-standard matter effect the perturbation theory is valid because it is expanded using small quantities $(U_s)_{ej}^*(U_s)_{ek} (j \neq k)$ and δV_{kl} .

5 Discussions and conclusions

In this paper we propose a perturbation theory which simplifies the problem of neutrino oscillation in the Earth. We perform analysis with three flavor of neutrinos. The perturbation theory is developed using the trajectory dependent average potential. The average potential is averaged along the neutrino trajectory in the Earth. So it depends on the trajectory of neutrino. The effect of non-constant density profile in the Earth, i.e. the non-adiabatic contribution, is carefully included in the first order of the perturbation theory. This perturbation theory is generalized using Eq. (12) for core crossing trajectories, i.e. for $L < 10690$ km. The problem of neutrino oscillation is substantially simplified using the perturbation theory presented in the present paper.

Using the perturbation theory we study neutrino oscillation in the Earth for cases with standard matter effect and with non-standard matter effect. It is shown that for both cases

the formulation presented gives a precise description of the neutrino oscillation. We study observables of flavor conversion and of time reversal asymmetry in neutrino oscillation. We find remarkable agreement between result of computation using our perturbation theory and that of numerical computation. This is a nontrivial agreement. To see that it is nontrivial one can try another perturbation theory. For the case with non-standard matter effect one may rewrite the hamiltonian as follows

$$H = H_1 + H_2, \quad H_1 = H_0 + \text{diag}\{\bar{V}_e, 0, 0\}, \quad H_2 = \begin{pmatrix} \delta V_e & V_{e\mu} & V_{e\tau} \\ V_{\mu e} & V_{\mu\mu} & V_{\mu\tau} \\ V_{\tau e} & V_{\tau\mu} & V_{\tau\tau} \end{pmatrix}.$$

One may first solve evolution problem of H_1 and treat H_2 as perturbation. The first order result of this perturbation theory is a good approximation for $E \lesssim 5$ GeV. Apparently it will not be good for higher energy neutrinos when non-standard matter effect becomes important. It is not able to reproduce the time reversal asymmetry in high energy region which is dominated by non-standard matter effect. To improve the perturbation theory one may go to higher order and make the description more complicated. Apparently this perturbation theory is not a good choice compared to one given in the present paper.

We also show the results of analytic computation, i.e. the computation using average potential but without non-adiabatic contribution. It is shown that the analytic result gives a very good approximation to the oscillation for $L \lesssim 6000$ km. It means that the problem of neutrino oscillation can be greatly simplified for $L \lesssim 6000$ km and $E \gtrsim 0.5$ GeV. For this range of parameter space matter effect in a fixed baseline is described by a constant potential term. For standard matter effect, in particular, it says that matter effect is a one parameter fit.

The perturbation theory given in the present paper is valid for $E \gtrsim 0.5$ GeV. Previous works [3, 4] cover the energy range $E \lesssim 30$ MeV. They are all energy ranges away from the 1 – 2 MSW resonance. It is a question to find a compact and simple formulation of neutrino oscillation in the Earth in the energy range $30 \text{ MeV} \lesssim E \lesssim 500 \text{ MeV}$ for which the 1 – 2 MSW resonance can happen in the Earth.

We perform analysis using the PREM density profile [16]. It is a symmetric density profile. We also assume ϵ_{kl} is constant in considering non-standard matter effect. These assumptions are not generally true. For more general density profile further research is needed to understand whether the perturbation theory is as good as analyzed in the present paper. We expect that this perturbation theory is still a good approximation in more general cases. Research on neutrino evolution with more general density profile will be presented in further publications.

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Appendix

In this appendix we show that in large V_e limit $\cos \theta_{13}^m$ is small. We write

$$H = H^x + \text{diag}\{V_e, 0, 0\}, \quad (41)$$

where

$$H^x = H_0, \quad (42)$$

for the case with standard matter effect, and

$$H^x = H_0 + \begin{pmatrix} 0 & V_{e\mu} & V_{e\tau} \\ V_{\mu e} & V_{\mu\mu} & V_{\mu\tau} \\ V_{\tau e} & V_{\tau\mu} & V_{\tau\tau} \end{pmatrix}, \quad (43)$$

for the case with non-standard matter effect.

We consider the eigenvalue problem of H :

$$H X = \lambda X, \quad X = (x_1, x_2, x_3)^T. \quad (44)$$

In the large V_e limit one of the engenvalues of H is roughly V_e . So at zeroth order $\lambda \approx V_e$. We solve the eigenvalue problem using perturbation in H^x/V_e :

$$\lambda = \lambda_0 + \lambda_1 + \lambda_2 + \dots, \quad X = X_0 + X_1 + X_2 + \dots \quad (45)$$

So to first order in H_x/V_e we get

$$\text{diag}\{V_e, 0, 0\} X_0 = \lambda_0 X_0, \quad (46)$$

$$H^x X_0 + \text{diag}\{V_e, 0, 0\} X_1 = \lambda_0 X_1 + \lambda_1 X_0. \quad (47)$$

We get from Eq. (46)

$$\lambda_0 = V_e, \quad X_0 = (1, 0, 0)^T. \quad (48)$$

Putting Eq. (48) into Eq. (47) we get

$$\lambda_1 = H_{ee}^x, \quad X_1 = (0, H_{\mu e}^x/V_e, H_{\tau e}^x/V_e)^T. \quad (49)$$

Writing $X = (\sin \theta_{13}^m e^{-i\delta_{13}^m}, \cos \theta_{13}^m \sin \theta_{23}^m, \cos \theta_{13}^m \cos \theta_{23}^m)^T$, we can get

$$\cos^2 \theta_{13}^m \approx (|H_{\mu e}^x|^2 + |H_{\tau e}^x|^2)/V_e^2. \quad (50)$$

Using Eq. (2) in the case with standard matter effect we can get

$$\cos \theta_{13}^m = \frac{|\Delta m_{31}^2|}{2E} \sqrt{1 - |(U_0)_{e3}|^2} |(U_0)_{e3}| = \frac{|\Delta m_{31}^2|}{4EV_e} \sin 2\theta_{13}. \quad (51)$$

In the case with non-standard matter effect there are two possibilities. One possibility is that effect of H_0 is larger than the non-standard matter effect. Hence, Eq. (51) holds and we have small $\cos \theta_{13}^m$ in large V_e limit. The other possibility is that non-standard matter effect is large enough to determine the mixing matrix. So we have

$$\cos \theta_{13}^m = \sqrt{|V_{\mu e}|^2 + |V_{\tau e}|^2} / V_e. \quad (52)$$

We conclude that $\cos \theta_{13}^m$ is a small quantity in large E limit for both the case with standard matter effect and the case with non-standard matter effect.

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